Math 31 – Homework 4

Due Wednesday, July 17

Note: Any problem labeled as "show" or "prove" should be written up as a formal proof, using complete sentences to convey your ideas.

Easier

1. Determine whether each of the following subsets is a subgroup of the given group. If not, state which of the subgroup axioms fails.

(a) The set of real numbers $\mathbb{R}$, viewed as a subset of the complex numbers $\mathbb{C}$ (under addition).

(b) The set $\pi\mathbb{Q}$ of rational multiples of π, as a subset of $\mathbb{R}$ (under addition).

(c) The set of $n \times n$ matrices with determinant 2, as a subset of $\mathrm{GL_n}(\mathbb{R})$.

(d) The set $\{i, m_1, m_2, m_3\} \subset D_3$ of reflections of the equilateral triangle, along with the identity transformation.

2. We proved in class that every subgroup of a cyclic group is cyclic. The following statement is almost the converse of this:

"Let G be a group. If every *proper* subgroup of G is cyclic, then G is cyclic."

Find a counterexample to the above statement.

3. [Saracino, #5.10] Prove that any subgroup of an abelian group is abelian.

Medium

4. [Saracino, #5.14] Let G be a group. If H and K are subgroups of G, show that $H \cap K$ is also a subgroup of G.

5. Let r and s be positive integers, and define

$$H = \{nr + ms : n, m \in \mathbb{Z}\}.$$

(a) Show that H is a subgroup of $\mathbb{Z}$.

(b) We saw in class that every subgroup of $\mathbb{Z}$ is cyclic. Therefore, $H = \langle d \rangle$ for some $d \in \mathbb{Z}$. What is this integer d? Prove that the d you've found is in fact a generator for H.

6. Let X be a set, and recall that S_X is the group consisting of the bijections from S to itself, with the binary operation given by composition of functions. (If X is finite, then S_X is just the symmetric group on n letters, where X has n elements.) Given $x_1 \in X$, define

$$H = \{f \in S_X : f(x_1) = x_1\}.$$

Show that $H \leq S_X$.

7. [Saracino, #5.22] Let G be a group. Define

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}.$$

In other words, the elements of $Z(G)$ are exactly those which commute with *every* element of G. Prove that $Z(G)$ is a subgroup of G, called the **center** of G.

8. Show that if H and K are subgroups of an *abelian* group G, then

$$\{hk : h \in H \text{ and } k \in K\}$$

is a subgroup of G.

9. [Saracino, #5.20] We will see in class that if p is a prime number, then the cyclic group $\mathbb{Z}_p$ has no proper subgroups as a consequence of Lagrange's theorem. This problem will have you investigate a "converse" to this result.

(a) If G is a finite group which has no proper subgroups (other than $\{e\}$), prove that G must be cyclic.

(b) Extend the result of (a) by showing that if G has no proper subgroups, then G is not only cyclic, but

$$|G| = p$$

for some prime number p.

Hard

10. [Saracino, #5.25 and 5.26] Let G be a group, and let H be a subgroup of G.

(a) Let a be some fixed element of G, and define

$$aHa^{-1} = \{aha^{-1} : h \in H\}.$$

This set is called the **conjugate** of H by a. Prove that aHa^{-1} is a subgroup of G.

(b) Define the **normalizer** of H in G to be

$$N(H) = \{a \in G : aHa^{-1} = H\}.$$

Prove that $N(H)$ is a subgroup of G.